MATH 320: PRACTICE PROBLEMS FOR THE FINAL AND SOLUTIONS

There will be eight problems on the final. The following are sample problems.

Problem 1. *Let $\mathcal{F}$ be the vector space of all real valued functions on the real line (i.e. $\mathcal{F} = \{f \mid f : \mathbb{R} \to \mathbb{R}\}$). Determine whether the following are subspaces of $\mathcal{F}$. Prove your answer.*

(1) $\{f \in \mathcal{F} \mid f(x) = -f(-x) \text{ for all } x\}$.
(2) $\{f \in \mathcal{F} \mid f(0) = 1\}$.
(3) $\{f \in \mathcal{F} \mid f(1) = 0\}$.

Solution 1. (1) $W = \{f \in \mathcal{F} \mid f(x) = -f(-x) \text{ for all } x\}$ is a subspace since the following hold:

- for all x, $\vec{0}(x) = 0 = -\vec{0}(-x)$, so $\vec{0} \in W$,
- if $f, g \in W$, then for all x, $(f+g)(x) = f(x) + g(x) = -f(-x) - g(-x) = -(f+g)(-x)$, so $f + g \in W$,
- if $f \in W$ and c is a scalar, then for all x, $(cf)(x) = cf(x) = c(-f(-x)) = -cf(-x) = -(cf)(-x)$, so $cf \in W$.

(2) $\{f \in \mathcal{F} \mid f(0) = 1\}$ is not a subspace since $\vec{0}$ is not in it.
(3) $S = \{f \in \mathcal{F} \mid f(1) = 0\}$ is a subspace since the following hold:

- $\vec{0}(1) = 0$, so $\vec{0} \in S$,
- if $f, g \in S$, then $(f+g)(1) = f(1) + g(1) = 0 + 0 = 0$, so $f + g \in S$,
- if $f \in S$ and c is a scalar, $(cf)(1) = cf(1) = c0 = 0$, so $cf \in S$.

Problem 2. *Suppose that $T : V \to V$. Recall that a subspace W is T-invariant if for all $x \in W$, we have that $T(x) \in W$.*

(1) *Prove that* $\text{ran}(T)$, *$ker(T)$ are both T-invariant.*
(2) *Suppose that W is a T-invariant subspace and $V = \text{ran}(T) \oplus W$. Show that $W \subset ker(T)$.*

Solution 2. For part (1), for any $x \in \text{ran}(T)$, we have that $T(x) \in \text{ran}(T)$, so the range is invariant. Also, if $x \in \ker(T)$, then $T(x) = 0 \in \ker(T)$, so the kernel is invariant.

For part (2), suppose that $x \in W$. Then $T(x) \in W$, since W is invariant. But also $T(x) \in \text{ran}(T)$. Since $V = \text{ran}(T) \oplus W$, we have that $\text{ran}(T) \cap W = \{0\}$. And since $T(x)$ is in that intersection, we have that $T(x) = 0$, so $x \in \ker(T)$. It follows that $W \subset \ker(T)$.

Problem 3. *Suppose that $T : V \to W$ is a linear transformation and $\{v_1, ..., v_n\}$ is a basis for V. Prove that T is an isomorphism if and only if $\{T(v_1), ..., T(v_n)\}$ is a basis for W.*

Solution 3. For the first direction, suppose that T is an isomorphism. We have to show that $\{T(v_1), ..., T(v_n)\}$ is a basis for W. First we will show that the vectors are linearly independent. Suppose that for some scalars $a_1, ..., a_n$, we have

$$\begin{gathered} a_1 T(v_1) + ... + a_n T(v_n) = 0 \\ \Rightarrow T(a_1 v_1 + ... a_n v_n) = 0 \\ \Rightarrow a_1 v_1 + ... + a_n v_n \in \ker(T). \end{gathered}$$

T is one-to-one, so it has a trivial kernel, so $a_1 v_1 + ... + a_n v_n = 0$. But $\{v_1, ..., v_n\}$ are linearly independent since they are a basis for V. So, $a_1 = ... = a_n = 0$. It follows that $\{T(v_1), ..., T(v_n)\}$ are linearly independent. Now, T is an isomorphism, and so $\dim(V) = \dim(W) = n$. Therefore $\{T(v_1), ..., T(v_n)\}$ is a basis for W.

For the other direction, suppose that $\{T(v_1), ..., T(v_n)\}$ is a basis for W. We have to show that T is onto and on-to-one. Let $y \in W$, then for some scalars $a_1, ..., a_n$, we have $y = a_1 T(v_1) + ... + a_n T(v_n) = T(a_1 v_1 + ... a_n v_n) \in \operatorname{ran}(T)$. Thus T is onto. To show that it is one-to-one, suppose that $T(x) = 0$, let $c_1, ..., c_n$ be such that $x = c_1 v_1 + ... + c_n v_n$ (here we use that $\{v_1, ..., v_n\}$ is a basis for V). Then $0 = T(x) = T(c_1 v_1 + ... + c_n v_n) = c_1 T(v_1) + ... + c_n T(v_n)$. But $\{T(v_1), ..., T(v_n)\}$ are linearly independent, so $c_1 = ... = c_n = 0$, and so $x = 0$. It follows that T is one-to-one.

Problem 4. *Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be $T(\langle x_1, x_2, x_3 \rangle) = \langle x_1 - x_2, x_2 - x_3, x_3 - x_1 \rangle$. Let $\beta = \{\langle 1, 1, 1 \rangle, \langle 1, 1, 0 \rangle, \langle 1, -1, 0 \rangle\}$ and let e be the standard basis for $\mathbb{R}^3$.*

1. *Find $[T]_e$.*
2. *Find $[T]_\beta$*
3. *Find an invertible matrix Q such that $[T]_\beta = Q^{-1}[T]_e Q$.*

Solution 4. $T(e_1) = \langle 1, 0, -1 \rangle$, $T(e_2) = \langle -1, 1, 0 \rangle$, $T(e_3) = \langle 0, -1, 1 \rangle$. So,

$$[T]_e = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{pmatrix}$$

For the second part, we have that:

- $T(\langle 1, 1, 1 \rangle) = \langle 0, 0, 0 \rangle$,
- $T(\langle 1, 1, 0 \rangle) = \langle 0, 1, -1 \rangle = -\langle 1, 1, 1 \rangle + \frac{3}{2}\langle 1, 1, 0 \rangle - \frac{1}{2}\langle 1, -1, 0 \rangle$,

- $T(\langle 1,-1,0\rangle) = \langle 2,-1,-1\rangle = -\langle 1,1,1\rangle + \frac{3}{2}\langle 1,1,0\rangle + \frac{3}{2}\langle 1,-1,0\rangle$

So,

$$[T]_\beta = \begin{pmatrix} 0 & -1 & -1 \\ 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & -\frac{1}{2} & \frac{3}{2} \end{pmatrix}$$

Finally, let

$$Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

Then $Q = [id]^e_\beta$ and so $[T]_\beta = Q^{-1}[T]_e Q$.

Problem 5. *Prove the theorem that a linear transformation is one-to-one if and only if it has a trivial kernel.*

Solution 5. For the left to right (and easier) direction, suppose that T is one-to-one. If $x \in \ker(T)$, then $T(x) = T(0) = 0$, so since T is one-to-one, we get $x = 0$. I.e. $\ker(T) = \{0\}$.

For the other direction, suppose that $\ker(T) = \{0\}$, and suppose that for some x, y, $T(x) = T(y)$. Then $T(x) - T(y) = T(x - y) = 0$, so $x - y \in \ker(T)$. By our assumption, it follows that $x - y = 0$, i.e. $x = y$. So, T is one-to-one.

Problem 6. *Determine if the following systems of linear equations are consistent*

(1)
$$\begin{aligned} &x + 2y + 3z = 1 \\ &x + y - z = 0 \\ &x + 2y + z = 3 \end{aligned}$$

(2)
$$\begin{aligned} &x + 2y - z = 1 \\ &2x + y + 2z = 3 \\ &x - 4y + 7z = 4 \end{aligned}$$

Solution 6. We can represent the system in part (1) as $A\vec{v} = \langle 1, 0, 3\rangle$, where $\vec{v} = \langle x, y, z\rangle$ and

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \end{pmatrix}$$

Row reducing A, we compute that the rank of A is 3, and so L_A is onto. Then $\langle 1, 0, 3\rangle$ is in its range, and so the system is consistent.

For the second system, we have $A\vec{v} = \langle 1, 3, 4 \rangle$, where $\vec{v} = \langle x, y, z \rangle$ and

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 1 & 2 \\ 1 & -4 & 7 \end{pmatrix}$$

Row reducing A, we compute that the rank of A is 2. On the other hand, setting $\vec{b} = \langle 1, 3, 4 \rangle$, we have that the rank of $[A|\vec{b}]$ is 3 (again we row reduce to compute that.) So, $\vec{b} \notin \operatorname{ran} L_A$. It follows that the system is inconsistent.

Problem 7. *Suppose that A, B are two $n \times n$ matrices. Prove that the rank of AB is less than or equal to the rank of B.*

Solution 7. First we note that by the dimension theorem, for any linear transformation $T : V \to W$ and subspace S of V, we have that $\dim(T''S) \leq \dim(S)$. Here $T''S$ denotes the image of S under T. Now, we have that $rank(AB) = rank(L_{AB}) = \dim(\operatorname{ran} L_{AB}) = \dim(\operatorname{ran}(L_A \circ L_B)) = \dim(L_A''(\operatorname{ran} L_B)) \leq \dim(\operatorname{ran} L_B) = rank(B)$.

Problem 8. *Suppose A, B are $n \times n$ matrices, such that B is obtained from A by multiplying a row of A by a nonzero scalar c. Prove that* $\det(B) = c\det(A)$. *(You can use the definition of determinant by expansion along any row or column.)*

Solution 8. If $n = 1$, then $A = (a)$, $B = (ca)$, and so $\det(B) = ca = c\det(A)$. Now, suppose that $n > 1$. Let $1 \leq k \leq n$ be such that the k-th row of A is multiplied by c to obtain B. Denote

$$A = (a_{ij})_{1 \leq i,j \leq n}, B = (b_{ij})_{1 \leq i,j \leq n}.$$

Then for all j, $b_{kj} = ca_{kj}$. Also, for $1 \leq i, j \leq n$ let A_{ij} and B_{ij} be the $n-1 \times n-1$ submatrices obtained by removing the i-th row and the j-th column of A and B respectively. Note that for all j, $A_{kj} = B_{kj}$. Then, expanding along the k-th row of B, we compute:

$$\det(B) = \Sigma_{1 \leq j \leq n} b_{kj}(-1)^{j+k} \det B_{kj} = \Sigma_{1 \leq j \leq n} ca_{kj}(-1)^{j+k} \det A_{kj} =$$

$$= c(\Sigma_{1 \leq j \leq n} a_{kj}(-1)^{j+k} \det A_{kj}) = c\det(A).$$

Problem 9. *Suppose M is an $n \times n$ matrix that can be written in the form*

$$M = \begin{pmatrix} A & B \\ 0 & I \end{pmatrix}$$

where A is a square matrix. Show that $\det(M) = \det(A)$.

Solution 9. We prove this by induction on k. If $k = 1$, then denote

$$M = \begin{pmatrix} a & B \\ 0 & I \end{pmatrix}$$

where a is a scalar. So, expanding by the first column, we get $\det(M) = a \det(I) = a = \det(A)$ as desired.

Now, suppose that $k > 1$ and the statement is true for $k-1$. Denote $A = (a_{ij})_{1 \leq i,j \leq k}$. Also, for $1 \leq i \leq k$, M_{i1} is the submatrix of M obtained by removing the i-th row and the first column. Then for each $i \leq k$, M_{i1} has the form:

$$\begin{pmatrix} A_{i1} & B_i \\ 0 & I \end{pmatrix}$$

where A_{i1} is the submatrix of A obtained by removing the i-th row and the first column of A, and B_i is the submatrix of B obtained by removing the i-th row of B. By the inductive hypothesis, we have that for each $i \leq k$, $\det(M_{i1}) = \det(A_{i1})$. Expanding by the first column, we get:

$$\det(M) = \Sigma_{1 \leq i \leq k} a_{i1} (-1)^{i+1} \det M_{i1} = \Sigma_{1 \leq i \leq k} a_{i1} (-1)^{i+1} \det A_{i1} = \det(A).$$

Problem 10. *A matrix A is called nilpotent if for some positive integer k, $A^k = 0$. Prove that if A is a nilpotent matrix, then A is not invertible.*

Solution 10. Here we will use the theorem that for any two matrices B, C, we have that $\det(BC) = \det(B)\det(C)$. Fix k such that $A^k = 0$. Then $0 = \det(A^k) = (\det(A))^k$. So, $\det(A) = 0$. It follows that A is not invertible.

Problem 11. *An $n \times n$ matrix A is called orthogonal if $AA^t = I_n$. Prove that if A is orthogonal, then $|\det A| = 1$.*

Solution 11. We will use the theorems that for any two matrices B, C, we have that $\det(BC) = \det(B)\det(C)$ and $\det(B^t) = \det(B)$.

Suppose that $AA^t = I_n$. Then $1 = \det(I_n) = \det(AA^t) = \det(A)\det(A^t) = \det(A)\det(A) = (\det(A))^2$. So $|\det(A)| = 1$.

Problem 12. *Let A be an $n \times n$ matrix. Prove that if A is diagonalizable, then so is A^t.*

Solution 12. Since A is diagonalizable, let C be the invertible matrix such that $C^{-1}AC = D$, where D is a diagonal matrix. Then $A = CDC^{-1}$, and so $A^t = (CDC^{-1})^t = (C^{-1})^t D^t C^t = (C^t)^{-1} D C^t$, and so $C^t A^t (C^t)^{-1} = D$. I.e. A^t is diagonalizable.

Problem 13. *Let*

$$A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

Show that A is diagonalizable over $\mathbb{R}$ and find an invertible matrix C such that $C^{-1}AC = D$ where D is diagonal.

Solution 13. The characteristic polynomial of A is $\lambda(1-\lambda)(2-\lambda)$. Setting this equal to zero, we get $\lambda = 0, 1, 2$. So, we have three eigenvalues. Since A is a 3×3 matrix and there are 3 eigenvalues, it follows that A must be diagonalizable.

To find the invertible matrix C, we have to find a basis of eigenvectors and use them as the column vectors of C.

- for $\lambda = 0$, we solve $A\mathbf{x} = \mathbf{0}$. $A\langle x_1, x_2, x_3\rangle = \langle x_1 + 3x_2, 2x_2 - x_3, 0\rangle$ and so $0 = x_1 + 3x_2 = 2x_2 - x_3$. So, $x_1 = -3x_2$ and $x_3 = 2x_2$, so $\mathbf{x} = c\langle -3, 1, 2\rangle$.
- for $\lambda = 1$, we solve $A\mathbf{x} = \mathbf{x}$. Then $x_1 = x_1 + 3x_2$, $x_2 = 2x_2 - x_3$, $x_3 = 0$. So, $x_2 = 0$ and $\mathbf{x} = c\langle 1, 0, 0\rangle$.
- for $\lambda = 2$, we solve $A\mathbf{x} = 2\mathbf{x}$. Then $2x_1 = x_1 + 3x_2$, $2x_2 = 2x_2 - x_3$, $2x_3 = 0$. So, $x_1 = 3x_2$ and $\mathbf{x} = c\langle 3, 1, 0\rangle$.

Now let

$$C = \begin{pmatrix} -3 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix}$$

$C^{-1}AC = D$, where

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Problem 14. *Let $T : V \to V$ be a linear transformation and let $x \in V$. Let W be the T-cyclic subspace of V generated by x. I.e. $W = Span(\{x, T(x), T^2(x), ...\})$.*

(1) *Show that W is T - invariant.*
(2) *Show that W is the smallest T-invariant subspace containing x (i.e. show that any T-invariant subspace that contains x, also contains W).*

Solution 14. For the first part of the problem, suppose that $y \in W$. Then for some scalars, $y = a_1x + a_2T(x) + ...a_{n+1}T^n(x)$, and so $T(y) = T(a_1x + a_2T(x) + ...a_{n+1}T^n(x)) = a_1T(x) + a_2T^2(x) + ...a_{n+1}T^{n+1}(x) \in W$. Thus W is T - invariant.

For the second part of the problem, suppose that S is a T-invariant subspace that contains x. First we show the following claim.

Claim 15. *For all $k \geq 0$, $T^k(x) \in S$.*

Proof. By induction on k. If $k = 0$, then $T^k(x) = x \in S$ by assumption. Now suppose that $T^k(x) \in S$. Then $T^{k+1}(x) = T(T^k(x)) \in S$ since S is T-invariant. □

Now for any $y \in W$, we know that for some scalars, $y = a_1 x + a_2 T(x) + ... a_{n+1} T^n(x)$ and since S is a subspace (i.e. closed under vector addition and scalar multiplication) we get that $y \in S$. So, $W \subset S$.

Problem 15. *Let*

$$A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$$

Use the Cayley-Hamilton theorem to show that $A^2 - 2A + 5I$ is the zero matrix.

Solution 16. The characteristic polynomial of A is $f(t) = \det(A - tI_2) = (1-t)^2 + 4 = t^2 - 2t + 5$. By the Cayley-Hamilton theorem, A satisfies its own characteristic polynomial. Therefore, $A^2 - 2A + 5I$ is the zero matrix.